Proofs of Selected Examples from The Theory of Partitions

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Chapter 1

Example 1

Proposition. The number of partitions of n in which each part appears two, three, or five times equals the number of partitions of n into parts congruent to 2, 3, 6, 9 or 10 modulo 12.

Lemma 1. A positive integer is congruent to 2, 3, 6, 9 or 10 modulo 12 iff it is an odd multiple of 2 or an odd multiple of 3.

Proof of Lemma 1. The odd residue classes modulo 12 are [1], [3], [5], [7], [9], and [11]. An odd multiple of 2 has the form $2m$, where m is odd. Thus the possible residue classes of $2m$ are

$$\begin{aligned}2[1] &= 2[7] = [2],\\ 2[3] &= 2[9] = [6],\\ 2[5] &= 2[11] = [10].\end{aligned}$$

An odd multiple of 3 has the form $3m$, where m is odd. Thus the possible residue classes of $3m$ are

$$\begin{aligned}3[1] &= 3[5] = 3[9] = [3],\\ 3[3] &= 3[7] = 3[11] = [9].\end{aligned}$$

Therefore, the odd multiples of 2 or 3 are exactly the integers congruent to 2, 3, 6, 9, or 10 modulo 12. ■

Proof of the Proposition. Let $p_\alpha(n)$ denote the number of partitions of n in which each part appears two, three, or five times. Let $p_\beta(n)$ denote the number of partitions of n into parts congruent to 2, 3, 6, 9, or 10 modulo 12. Thus

$$\sum_{n\geq 0} p_\alpha(n)q^n = \prod_{n\geq 1}\left(1+q^{2n}+q^{3n}+q^{5n}\right),$$

and by Lemma 1

$$\sum_{n\geq 0} p_\beta(n)q^n = \prod_{n\geq 1}\frac{1}{(1-q^{2(2n-1)})(1-q^{3(2n-1)})}.$$

Finally

$$\begin{aligned}\sum_{n\geq 0} p_\beta(n)q^n &= \prod_{n\geq 1}\frac{1-q^{4n}}{1-q^{2n}}\cdot\frac{1-q^{6n}}{1-q^{3n}}\\ &= \prod_{n\geq 1}\left(1+q^{2n}\right)\left(1+q^{3n}\right)\\ &= \prod_{n\geq 1}\left(1+q^{2n}+q^{3n}+q^{5n}\right)\\ &= \sum_{n\geq 0} p_\alpha(n)q^n.\end{aligned}$$

■

Proof of the Proposition (q-Pochhammer). Let $p_\alpha(n)$ denote the number of partitions of n in which each part appears two, three, or five times. Let $p_\beta(n)$ denote the number of partitions of n into parts congruent to 2, 3, 6, 9, or 10 modulo 12. Thus

$$\sum_{n\geq 0} p_\alpha(n)q^n = (-q^2;q^2)_\infty(-q^3;q^3)_\infty,$$

and by Lemma 1

$$\sum_{n\geq 0} p_\beta(n)q^n = \frac{1}{(q^2;q^4)_\infty(q^3;q^6)_\infty}.$$

Finally

$$\begin{aligned}\sum_{n\geq 0} p_\beta(n)q^n &= \frac{(q^4;q^4)_\infty(q^6;q^6)_\infty}{(q^2;q^2)_\infty(q^3;q^3)_\infty}\\ &= (-q^2;q^2)_\infty(-q^3;q^3)_\infty\\ &= \sum_{n\geq 0} p_\alpha(n)q^n.\end{aligned}$$

■

Example 2

Proposition. The number of partitions of n in which only odd parts may be repeated, here denoted $p_o(n)$, equals the number of partitions of n in which no part is repeated more than three times, here denoted $p_3(n)$.

Proof.

$$\begin{aligned}
\sum_{n\geq 0} p_o(n)q^n &= \prod_{n\geq 1}(1+q^{2n})\left(\frac{1}{1-q^{2n-1}}\right) \\
&= \prod_{n\geq 1}\left(\frac{1-q^{4n}}{1-q^{2n}}\right)\left(\frac{1}{1-q^{2n-1}}\right) \\
&= \frac{\prod_{n\geq 1}(1-q^{4n})}{\prod_{n\geq 1}(1-q^{2n})(1-q^{2n-1})} \\
&= \prod_{n\geq 1}\frac{1-q^{4n}}{1-q^n} \\
&= \sum_{n\geq 0} p_3(n)q^n. \qquad \blacksquare
\end{aligned}$$

Proof (q-Pochhammer).

$$\begin{aligned}
\sum_{n\geq 0} p_o(n)q^n &= (-q^2;q^2)_\infty (q;q^2)_\infty^{-1} \\
&= \frac{(q^4;q^4)_\infty}{(q^2;q^2)_\infty (q;q^2)_\infty} \\
&= \frac{(q^4;q^4)_\infty}{(q;q)_\infty} \\
&= \sum_{n\geq 0} p_3(n)q^n.
\end{aligned}$$

$\blacksquare$

Example 3

Proposition. The number of partitions of n in which only parts $\not\equiv 0 (\operatorname{mod} 2^m)$ may be repeated, here denoted $p_\alpha(n)$, equals the number of partitions of n in which no part appears more than $2^{m+1} - 1$ times, here denoted $p_\beta(n)$.

Proof.

$$\begin{aligned}
\sum_{n\geq 0} p_\alpha(n) q^n &= \prod_{n\geq 1} \left(1 + q^{2^m n}\right) \left(\frac{1 - q^{2^m n}}{1 - q^n}\right) \\
&= \prod_{n\geq 1} \left(1 + q^{2^m n}\right) \left(1 + q^n + \cdots + q^{(2^m - 1)n}\right) \\
&= \prod_{n\geq 1} \left(1 + q^n + \cdots + q^{(2^{m+1} - 1)n}\right) \\
&= \prod_{n\geq 1} \frac{1 - q^{2^{m+1} n}}{1 - q^n} \\
&= \sum_{n\geq 0} p_\beta(n) q^n.
\end{aligned}$$

■

Proof (q-Pochhammer).

$$\begin{aligned}
\sum_{n\geq 0} p_\alpha(n) q^n &= (-q^{2^m}, q^{2^m}; q^{2^m})_\infty (q; q)_\infty^{-1} \\
&= \frac{(q^{2^{m+1}}; q^{2^{m+1}})_\infty}{(q; q)_\infty} \\
&= \sum_{n\geq 0} p_\beta(n) q^n.
\end{aligned}$$

■