Personal Notes on Partition Theory

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Table of contents

My Propositions

Proposition 1. The number of partitions of n in which parts $\lambda_1, \lambda_2, \ldots, \lambda_m$ appear at least once equals the number of partitions of $n - \sum_{i=1}^{m} \lambda_i$.

> **Note**
>
> Proposition 2 is directly inspired by Chapter 1 examples 1 and 2 from The Theory of Partitions by George E. Andrews.

Proposition 2. The following are equal: The number of partitions of n in which multiples of x are distinct, here denoted $p_\alpha(n)$; the number of partitions of n in which no part is divisible by $2x$, here denoted $p_\beta(n)$; the number of partitions of n in which no part appears more than $2x-1$ times, here denoted $p_\gamma(n)$.

Proof.

$$\begin{aligned}\sum_{n\geq 0} p_\alpha(n)q^n &= \prod_{n\geq 1}\left(1+q^{xn}\right)\frac{1-q^{xn}}{1-q^n} \\ &= \frac{(q^{2x};q^{2x})_\infty}{(q;q)_\infty} \\ &= \sum_{n\geq 0} p_\beta(n)q^n.\end{aligned}$$

By Glashier's Theorem (Theorem 1) $p_\beta(n) = p_\gamma(n)$, thus $p_\alpha(n) = p_\gamma(n)$. ■

Theorems

Theorem 1 (Glashier's Theorem)**.** The number of partitions of n in which no part appears more than d times, here denoted $p_\alpha(n)$, equals the number of partitions of n in which no part is a divisible by $d+1$, here denoted $p_\beta(n)$.

Proof.

$$\begin{aligned}
\sum_{n\geq 0} p_\alpha(n) q^n &= \prod_{n\geq 1} \left(1 + q^n + \cdots + q^{dn}\right) \\
&= \prod_{n\geq 1} \frac{1-q^{(d+1)n}}{1-q^n} \\
&= \frac{(1-q^{d+1})(1-q^{2(d+1)})\cdots}{(1-q)\cdots(1-q^{d+1})\cdots(1-q^{2(d+1)})\cdots} \\
&= \prod_{n\geq 1,\, d+1\nmid n} \frac{1}{1-q^n} \\
&= \sum_{n\geq 0} p_\beta(n) q^n.
\end{aligned}$$

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