

Proofs of Selected Examples from *The Theory of Partitions*

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Contents

1	Chapter 1	1
	Example 1	1
	Example 2	3
	Example 3	4
	Example 4	5

1 Chapter 1

Example 1

Proposition. The number of partitions of n in which each part appears two, three, or five times equals the number of partitions of n into parts congruent to 2, 3, 6, 9, or 10 modulo 12.

Lemma 1. A positive integer is congruent to 2, 3, 6, 9, or 10 modulo 12 iff it is an odd multiple of 2 or an odd multiple of 3.

Proof of Lemma 1. The odd residue classes modulo 12 are $[1]$, $[3]$, $[5]$, $[7]$, $[9]$, and $[11]$. An odd multiple of 2 has the form $2m$, where m is odd. Thus, the possible residue classes of $2m$ are

$$\begin{aligned}2[1] &= 2[7] = [2], \\2[3] &= 2[9] = [6], \\2[5] &= 2[11] = [10].\end{aligned}$$

An odd multiple of 3 has the form $3m$, where m is odd. Thus, the possible residue classes of $3m$ are

$$\begin{aligned}3[1] &= 3[5] = 3[9] = [3], \\3[3] &= 3[7] = 3[11] = [9].\end{aligned}$$

Therefore, the odd multiples of two or three are exactly the integers congruent to 2, 3, 6, 9, or 10 modulo 12. $\blacksquare$

Proof of the Proposition. Let $p_\alpha(n)$ denote the number of partitions of n in which each part appears two, three, or five times. Let $p_\beta(n)$ denote the number of partitions of n into parts congruent to 2, 3, 6, 9, or 10 modulo 12. Thus

$$\sum_{n \geq 0} p_\alpha(n) q^n = \prod_{n \geq 1} (1 + q^{2n} + q^{3n} + q^{5n}),$$

and by Lemma 1

$$\sum_{n \geq 0} p_\beta(n) q^n = \prod_{n \geq 1} \frac{1}{(1 - q^{2(2n-1)})(1 - q^{3(2n-1)})}.$$

Finally

$$\begin{aligned}
\sum_{n \geq 0} p_{\beta}(n) q^n &= \prod_{n \geq 1} \frac{1 - q^{4n}}{1 - q^{2n}} \cdot \frac{1 - q^{6n}}{1 - q^{3n}} \\
&= \prod_{n \geq 1} (1 + q^{2n}) (1 + q^{3n}) \\
&= \prod_{n \geq 1} (1 + q^{2n} + q^{3n} + q^{5n}) \\
&= \sum_{n \geq 0} p_{\alpha}(n) q^n.
\end{aligned}$$

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Proof of the Proposition (q-Pochhammer). Let $p_{\alpha}(n)$ denote the number of partitions of n in which each part appears two, three, or five times. Let $p_{\beta}(n)$ denote the number of partitions of n into parts congruent to 2, 3, 6, 9, or 10 modulo 12. Thus

$$\sum_{n \geq 0} p_{\alpha}(n) q^n = (-q^2; q^2)_{\infty} (-q^3; q^3)_{\infty},$$

and by Lemma 1

$$\sum_{n \geq 0} p_{\beta}(n) q^n = \frac{1}{(q^2; q^4)_{\infty} (q^3; q^6)_{\infty}}.$$

Finally

$$\begin{aligned}
\sum_{n \geq 0} p_{\beta}(n) q^n &= \frac{(q^4; q^4)_{\infty} (q^6; q^6)_{\infty}}{(q^2; q^2)_{\infty} (q^3; q^3)_{\infty}} \\
&= (-q^2; q^2)_{\infty} (-q^3; q^3)_{\infty} \\
&= \sum_{n \geq 0} p_{\alpha}(n) q^n.
\end{aligned}$$

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Example 2

Proposition. The number of partitions of n in which only odd parts may be repeated, here denoted $p_o(n)$, equals the number of partitions of n in which no part is repeated more than three times, here denoted $p_3(n)$.

Proof.

$$\begin{aligned}
\sum_{n \geq 0} p_o(n)q^n &= \prod_{n \geq 1} (1 + q^{2n}) \left(\frac{1}{1 - q^{2n-1}} \right) \\
&= \prod_{n \geq 1} \left(\frac{1 - q^{4n}}{1 - q^{2n}} \right) \left(\frac{1}{1 - q^{2n-1}} \right) \\
&= \frac{\prod_{n \geq 1} (1 - q^{4n})}{\prod_{n \geq 1} (1 - q^{2n})(1 - q^{2n-1})} \\
&= \prod_{n \geq 1} \frac{1 - q^{4n}}{1 - q^n} \\
&= \sum_{n \geq 0} p_3(n)q^n.
\end{aligned}$$

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Proof (q-Pochhammer).

$$\begin{aligned}
\sum_{n \geq 0} p_o(n)q^n &= (-q^2; q^2)_\infty (q; q^2)_\infty^{-1} \\
&= \frac{(q^4; q^4)_\infty}{(q^2; q^2)_\infty (q; q^2)_\infty} \\
&= \frac{(q^4; q^4)_\infty}{(q; q)_\infty} \\
&= \sum_{n \geq 0} p_3(n)q^n.
\end{aligned}$$

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Example 3

Proposition. The number of partitions of n in which only parts $\not\equiv 0 \pmod{2^m}$ may be repeated, here denoted $p_\alpha(n)$, equals the number of partitions of n in which no part appears more than $2^{m+1} - 1$ times, here denoted $p_\beta(n)$.

Proof.

$$\begin{aligned}
\sum_{n \geq 0} p_\alpha(n) q^n &= \prod_{n \geq 1} (1 + q^{2^m n}) \left(\frac{1 - q^{2^m n}}{1 - q^n} \right) \\
&= \prod_{n \geq 1} (1 + q^{2^m n}) (1 + q^n + \cdots + q^{(2^m - 1)n}) \\
&= \prod_{n \geq 1} (1 + q^n + \cdots + q^{(2^{m+1} - 1)n}) \\
&= \prod_{n \geq 1} \frac{1 - q^{2^{m+1} n}}{1 - q^n} \\
&= \sum_{n \geq 0} p_\beta(n) q^n.
\end{aligned}$$

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Proof (q-Pochhammer).

$$\begin{aligned}
\sum_{n \geq 0} p_\alpha(n) q^n &= (-q^{2^m}, q^{2^m}; q^{2^m})_\infty (q; q)_\infty^{-1} \\
&= \frac{(q^{2^{m+1}}; q^{2^{m+1}})_\infty}{(q; q)_\infty} \\
&= \sum_{n \geq 0} p_\beta(n) q^n.
\end{aligned}$$

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Example 4

Proposition. The number of partitions of n with unique smallest part and largest part at most twice the smallest part, here denoted $p_\alpha(n)$, equals the number of partitions of n in which the largest part is odd and the smallest part is larger than half the largest part, here denoted $p_\beta(n)$.

Scratchwork

Let

$$a_n(q) = \frac{q^n}{(q^{n+1}; q)_n}, \quad b_n(q) = \frac{q^{2n-1}}{(q^n; q)_n}, \quad (1)$$

And let

$$A_m(q) = \sum_{n=1}^{2m-1} a_n(q), \quad B_m(q) = \sum_{n=1}^m b_n(q). \quad (2)$$

Then

$$\sum_{n \geq 1} p_\alpha(n) q^n = \sum_{n \geq 1} a_n(q) = \lim_{m \rightarrow \infty} A_m(q) = A(q), \quad (3)$$

And

$$\sum_{n \geq 1} p_\beta(n) q^n = \sum_{n \geq 1} b_n(q) = \lim_{m \rightarrow \infty} B_m(q) = B(q). \quad (4)$$

Conjecture 1. For all $m \in \mathbb{Z}_+$,

$$A_m(q) \equiv B_m(q) \pmod{q^{2m}}. \quad (5)$$

Let

$$\rho_r(n, m) = \frac{1}{m} \sum_{j=0}^{m-1} e^{2\pi i j(n-r)/m} = \begin{cases} 1 & \text{if } n \equiv r \pmod{m} \\ 0 & \text{otherwise} \end{cases} \quad (6)$$

so that, with $x_i \geq 0$, the coefficient extraction operations are

$$[q^N]a_v(q) = \sum_{x_1+x_2+\dots+x_v=N-v} \prod_{i=1}^v \rho_0(x_i, v+i), \quad (7)$$

And

$$[q^N]b_v(q) = \sum_{x_1+x_2+\dots+x_v=N-(2v-1)} \prod_{i=1}^v \rho_0(x_i, v+i-1). \quad (8)$$

Then we have

$$[q^N]A_m(q) = \sum_{v=1}^{2m-1} [q^N]a_v(q), \quad (9)$$

And

$$[q^N]B_m(q) = \sum_{v=1}^m [q^N]b_v(q). \quad (10)$$

Conjecture 2. For all $0 \leq N < 2m$

$$[q^N]A_m(q) = [q^N]B_m(q). \quad (11)$$

Conjecture 1 and Conjecture 2 are equivalent.

Other Observations

For $m > 2$

$$\frac{A_m(q) - B_m(q)}{q^{2m}} = \frac{1 - q^{2\lceil m/2 \rceil + 2}}{1 - q^2} + F(q) \quad (12)$$

Where $F(q)$ is some power series in which the lowest-degree term is of a degree greater than m .

Expressed in terms of ${}_r\phi_s$ basic hypergeometric series

$$\begin{aligned} \sum_{n \geq 1} p_\alpha(n) q^n &= {}_4\phi_3 \left[\begin{matrix} q, 0, 0, 0 \\ -q, q^{\frac{1}{2}}, -q^{\frac{1}{2}} \end{matrix}; q, q \right] - 1, \\ \sum_{n \geq 1} p_\beta(n) q^n &= \frac{q}{1 - q} {}_4\phi_3 \left[\begin{matrix} q, 0, 0, 0 \\ -q, q^{\frac{3}{2}}, -q^{\frac{3}{2}} \end{matrix}; q, q^2 \right], \end{aligned} \quad (13)$$

Where ${}_r\phi_s$ is defined as

$${}_r\phi_s \left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, z \right] = \sum_{n \geq 0} \frac{(a_1, a_2, \dots, a_r; q)_n}{(q, b_1, \dots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-r} z^n.$$