Proofs of Selected Examples from *The Theory of Partitions*

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Contents

1 Chapter 1

Example 1

Proposition. The number of partitions of n in which each part appears two, three, or five times equals the number of partitions of n into parts congruent to 2, 3, 6, 9, or 10 modulo 12.
Lemma 1. A positive integer is congruent to 2, 3, 6, 9, or 10 modulo 12 iff it is an odd multiple of 2 or an odd multiple of 3.

Proof of Lemma 1. The odd residue classes modulo 12 are [1], [3], [5], [7], [9], and [11]. An odd multiple of 2 has the form $2m$, where m is odd. Thus, the possible residue classes of $2m$ are

$$\begin{aligned} 2[1] &= 2[7] = [2], \\ 2[3] &= 2[9] = [6], \\ 2[5] &= 2[11] = [10]. \end{aligned}$$

An odd multiple of 3 has the form $3m$, where m is odd. Thus, the possible residue classes of $3m$ are

$$\begin{aligned} 3[1] &= 3[5] = 3[9] = [3], \\ 3[3] &= 3[7] = 3[11] = [9]. \end{aligned}$$

Therefore, the odd multiples of two or three are exactly the integers congruent to 2, 3, 6, 9, or 10 modulo 12. ■

Proof of the Proposition. Let $p_\alpha(n)$ denote the number of partitions of n in which each part appears two, three, or five times. Let $p_\beta(n)$ denote the number of partitions of n into parts congruent to 2, 3, 6, 9, or 10 modulo 12. Thus

$$\sum_{n\geq 0} p_\alpha(n)q^n = \prod_{n\geq 1}\left(1+q^{2n}+q^{3n}+q^{5n}\right),$$

and by Lemma 1

$$\sum_{n\geq 0} p_\beta(n)q^n = \prod_{n\geq 1}\frac{1}{(1-q^{2(2n-1)})(1-q^{3(2n-1)})}.$$

Finally

$$\begin{aligned}\sum_{n\geq 0} p_\beta(n)q^n &= \prod_{n\geq 1} \frac{1-q^{4n}}{1-q^{2n}} \cdot \frac{1-q^{6n}}{1-q^{3n}} \\ &= \prod_{n\geq 1} \left(1+q^{2n}\right)\left(1+q^{3n}\right) \\ &= \prod_{n\geq 1} \left(1+q^{2n}+q^{3n}+q^{5n}\right) \\ &= \sum_{n\geq 0} p_\alpha(n)q^n.\end{aligned}$$

■

Proof of the Proposition (q-Pochhammer). Let $p_\alpha(n)$ denote the number of partitions of n in which each part appears two, three, or five times. Let $p_\beta(n)$ denote the number of partitions of n into parts congruent to 2, 3, 6, 9, or 10 modulo 12. Thus

$$\sum_{n\geq 0} p_\alpha(n)q^n = (-q^2;q^2)_\infty(-q^3;q^3)_\infty,$$

and by Lemma 1

$$\sum_{n\geq 0} p_\beta(n)q^n = \frac{1}{(q^2;q^4)_\infty(q^3;q^6)_\infty}.$$

Finally

$$\begin{aligned}\sum_{n\geq 0} p_\beta(n)q^n &= \frac{(q^4;q^4)_\infty(q^6;q^6)_\infty}{(q^2;q^2)_\infty(q^3;q^3)_\infty} \\ &= (-q^2;q^2)_\infty(-q^3;q^3)_\infty \\ &= \sum_{n\geq 0} p_\alpha(n)q^n.\end{aligned}$$

■

Example 2

Proposition. The number of partitions of n in which only odd parts may be repeated, here denoted $p_o(n)$, equals the number of partitions of n in which no part is repeated more than three times, here denoted $p_3(n)$.

Proof.

$$\begin{aligned}
\sum_{n\geq 0} p_o(n)q^n &= \prod_{n\geq 1} \left(1+q^{2n}\right)\left(\frac{1}{1-q^{2n-1}}\right) \\
&= \prod_{n\geq 1} \left(\frac{1-q^{4n}}{1-q^{2n}}\right)\left(\frac{1}{1-q^{2n-1}}\right) \\
&= \frac{\prod_{n\geq 1}(1-q^{4n})}{\prod_{n\geq 1}(1-q^{2n})(1-q^{2n-1})} \\
&= \prod_{n\geq 1} \frac{1-q^{4n}}{1-q^n} \\
&= \sum_{n\geq 0} p_3(n)q^n.
\end{aligned}$$

■

Proof (q-Pochhammer).

$$\begin{aligned}
\sum_{n\geq 0} p_o(n)q^n &= (-q^2;q^2)_\infty (q;q^2)_\infty^{-1} \\
&= \frac{(q^4;q^4)_\infty}{(q^2;q^2)_\infty (q;q^2)_\infty} \\
&= \frac{(q^4;q^4)_\infty}{(q;q)_\infty} \\
&= \sum_{n\geq 0} p_3(n)q^n.
\end{aligned}$$

■

Example 3

Proposition. The number of partitions of n in which only parts $\not\equiv 0 \pmod{2^m}$ may be repeated, here denoted $p_\alpha(n)$, equals the number of partitions of n in which no part appears more than $2^{m+1}-1$ times, here denoted $p_\beta(n)$.

Proof.

$$\begin{aligned}
\sum_{n\geq 0} p_\alpha(n)q^n &= \prod_{n\geq 1}\left(1+q^{2^m n}\right)\left(\frac{1-q^{2^m n}}{1-q^n}\right) \\
&= \prod_{n\geq 1}\left(1+q^{2^m n}\right)\left(1+q^n+\cdots+q^{(2^m-1)n}\right) \\
&= \prod_{n\geq 1}\left(1+q^n+\cdots+q^{(2^{m+1}-1)n}\right) \\
&= \prod_{n\geq 1}\frac{1-q^{2^{m+1}n}}{1-q^n} \\
&= \sum_{n\geq 0} p_\beta(n)q^n.
\end{aligned}$$

■

Proof (q-Pochhammer).

$$\begin{aligned}
\sum_{n\geq 0} p_\alpha(n)q^n &= (-q^{2^m}, q^{2^m}; q^{2^m})_\infty (q;q)_\infty^{-1} \\
&= \frac{(q^{2^{m+1}}; q^{2^{m+1}})_\infty}{(q;q)_\infty} \\
&= \sum_{n\geq 0} p_\beta(n)q^n.
\end{aligned}$$

■

Example 4

Proposition. The number of partitions of n with unique smallest part and largest part at most twice the smallest part, here denoted $p_\alpha(n)$, equals the number of partitions of n in which the largest part is odd and the smallest part is larger than half the largest part, here denoted $p_\beta(n)$.

Scratchwork

Let

$$a_n(q) = \frac{q^n}{(q^{n+1};q)_n}, \quad b_n(q) = \frac{q^{2n-1}}{(q^n;q)_n}, \tag{1}$$

And let

$$A_m(q) = \sum_{n=1}^{2m-1} a_n(q), \quad B_m(q) = \sum_{n=1}^{m} b_n(q). \tag{2}$$

Then

$$\sum_{n\geq 1} p_\alpha(n)q^n = \sum_{n\geq 1} a_n(q) = \lim_{m\to\infty} A_m(q) = A(q), \tag{3}$$

And

$$\sum_{n\geq 1} p_\beta(n)q^n = \sum_{n\geq 1} b_n(q) = \lim_{m\to\infty} B_m(q) = B(q). \tag{4}$$

Conjecture 1. *For all $m \in \mathbb{Z}_+$,*

$$A_m(q) \equiv B_m(q) \pmod{q^{2m}}. \tag{5}$$

Let

$$\rho_r(n,m) = \frac{1}{m}\sum_{j=0}^{m-1} e^{2\pi i j(n-r)/m} = \begin{cases} 1 & \text{if } n \equiv r \pmod{m} \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

so that, with $x_i \geq 0$, the coefficient extraction operations are

$$[q^N]a_v(q) = \sum_{x_1+x_2+\cdots+x_v=N-v} \prod_{i=1}^{v} \rho_0(x_i, v+i), \tag{7}$$

And

$$[q^N]b_v(q) = \sum_{x_1+x_2+\cdots+x_v=N-(2v-1)} \prod_{i=1}^{v} \rho_0(x_i, v+i-1). \tag{8}$$

Then we have

$$[q^N]A_m(q) = \sum_{v=1}^{2m-1} [q^N]a_v(q), \tag{9}$$

And

$$[q^N]B_m(q) = \sum_{v=1}^{m} [q^N]b_v(q). \tag{10}$$

Conjecture 2. *For all* $0 \leq N < 2m$

$$[q^N]A_m(q) = [q^N]B_m(q). \tag{11}$$

Conjecture 1 and Conjecture 2 are equivalent.

Other Observations

For $m > 2$

$$\frac{A_m(q) - B_m(q)}{q^{2m}} = \frac{1 - q^{2\lceil m/2 \rceil + 2}}{1 - q^2} + F(q) \tag{12}$$

Where $F(q)$ is some power series in which the lowest-degree term is of a degree greater than m.

Expressed in terms of ${}_r\phi_s$ basic hypergeometric series

$$\sum_{n \geq 1} p_\alpha(n) q^n = {}_4\phi_3 \left[\begin{matrix} q, 0, 0, 0 \\ -q, q^{\frac{1}{2}}, -q^{\frac{1}{2}} \end{matrix} ; q, q \right] - 1, \tag{13}$$

$$\sum_{n \geq 1} p_\beta(n) q^n = \frac{q}{1-q} \, {}_4\phi_3 \left[\begin{matrix} q, 0, 0, 0 \\ -q, q^{\frac{3}{2}}, -q^{\frac{3}{2}} \end{matrix} ; q, q^2 \right],$$

Where ${}_r\phi_s$ is defined as

$${}_r\phi_s \left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} ; q, z \right] = \sum_{n \geq 0} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(q, b_1, \ldots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-r} z^n.$$