Exercises with q-Shifted Factorials

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2026.06.20

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Lecture Notes For An Introductory Minicourse on q-Series

Exercise 1.1

Verify the inversion identity.

Note that $0 + 1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2} = \binom{n}{2}$.

$$\begin{aligned}
(a;q)_n &= (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}) \\
&= (1-a)(q^{-1}-a)(q^{-2}-a)\cdots(q^{-(n-1)}-a)\, q^{\binom{n}{2}} \\
&= (1-a^{-1})(1-a^{-1}q^{-1})\cdots(1-a^{-1}q^{-(n-1)})(-a)^n q^{\binom{n}{2}} \\
&= (a^{-1};q^{-1})_n(-a)^n q^{\binom{n}{2}}
\end{aligned}$$

In parts (ii) through (v), k and n are nonnegative integers.

(i)

$$\begin{aligned}
(a;q)_n &= (1-a)(1-aq)\cdots(1-aq^{n-1}) \\
&= \frac{(1-a)(1-aq)\cdots(1-aq^{n-1})(1-aq^n)\cdots}{(1-aq^n)(1-aq^{n+1})\cdots} \\
&= \frac{(a;q)_\infty}{(aq^n;q)_\infty}
\end{aligned}$$

(ii)

$$\begin{aligned}
(a;q)_{n+k} &= (1-a)\cdots(1-aq^{n-1})\cdot(1-aq^n)\cdots(1-aq^{n+k-1}) \\
&= (a;q)_n(aq^n;q)_k \\
&= (1-a)\cdots(1-aq^{k-1})\cdot(1-aq^k)\cdots(1-aq^{k+n-1}) \\
&= (a;q)_k(aq^k;q)_n
\end{aligned}$$

(iii)

$$(aq^n;q)_k = \frac{(a;q)_n(aq^n;q)_k}{(a;q)_n} = \frac{(a;q)_k(aq^k;q)_n}{(a;q)_n}$$

(iv)

By identity (ii),

$$(a;q)_{k-n} = (a;q)_{-n}(aq^{-n};q)_k = (a;q)_k(aq^k;q)_{-n}.$$

We begin by writing

$$(aq^{-n};q)_k = \frac{(a;q)_{k-n}}{(a;q)_{-n}} = \frac{(a;q)_k(aq^k;q)_{-n}}{(a;q)_{-n}}.$$

Using the definition $(a;q)_{-n} = (aq^{-n};q)_n^{-1}$, we simplify this to

$$(aq^{-n};q)_k = \frac{(a;q)_k(aq^{-n};q)_n}{(aq^{k-n};q)_n}.$$

We can simplify further:

$$\begin{aligned}(aq^{-n};q)_n &= \prod_{j=0}^{n-1}(1-aq^{-n}q^j) = \prod_{j=1}^{n}(1-aq^{-j}) \\ &= \prod_{j=1}^{n}(-aq^{-j})(1-a^{-1}q^j) = (-a)^n q^{-\binom{n+1}{2}}(a^{-1}q;q)_n,\end{aligned}$$

and

$$\begin{aligned}(aq^{k-n};q)_n &= \prod_{j=0}^{n-1}(1-aq^{k-n}q^j) = \prod_{j=1}^{n}(1-aq^{k-j}) \\ &= \prod_{j=1}^{n}(-aq^{k-j})(1-a^{-1}q^{j-k}) = (-a)^n q^{kn} q^{-\binom{n+1}{2}}(a^{-1}q^{1-k};q)_n.\end{aligned}$$

Thus

$$\begin{aligned}(aq^{-n};q)_k &= \frac{(a;q)_k(-a)^n q^{-\binom{n+1}{2}}(a^{-1}q;q)_n}{(-a)^n q^{kn} q^{-\binom{n+1}{2}}(a^{-1}q^{1-k};q)_n} \\ &= \frac{(a;q)_k(a^{-1}q;q)_n}{(a^{-1}q^{1-k};q)_n}q^{-kn}.\end{aligned}$$

(v)

$$\begin{aligned}\frac{1-aq^{2n}}{1-a} = \frac{(aq^2;q^2)_n}{(a;q^2)_n} &= \frac{(1-aq^2)\cdots(1-aq^{2n})}{(1-a)\cdots(1-aq^{2n-2})} \\ &= \frac{(1-a^{\frac{1}{2}}q)(1+a^{\frac{1}{2}}q)\cdots(1-a^{\frac{1}{2}}q^n)(1+a^{\frac{1}{2}}q^n)}{(1-a^{\frac{1}{2}})(1+a^{\frac{1}{2}})\cdots(1-a^{\frac{1}{2}}q^{n-1})(1+a^{\frac{1}{2}}q^{n-1})} \\ &= \frac{(qa^{\frac{1}{2}};q)_n(-qa^{\frac{1}{2}};q)_n}{(a^{\frac{1}{2}};q)_n(-a^{\frac{1}{2}};q)_n}.\end{aligned}$$

Basic Hypergeometric Series

Exercise 1.1

Verify the identities (1.2.30)-(1.2.40); k and n are integers.

(1.2.30)

In the case that n is a positive integer, See exercise 1.1(i) on page 1. If $n=-m$, $m \in \mathbb{Z}_+$ then

$$(a;q)_{-m} = \frac{1}{(aq^{-m};q)_m} = \frac{(a;q)_\infty}{(aq^{-m};q)_\infty}.$$

(1.2.31)

Note that $0+1+2+\cdots+(n-1) = \frac{n(n-1)}{2} = \binom{n}{2}$.

$$\begin{aligned}(a^{-1}q^{1-n};q)_n &= \prod_{j=0}^{n-1}(1-a^{-1}q^{1-n+j}) = \prod_{j=0}^{n-1}(1-a^{-1}q^{-j})\\ &= \prod_{j=0}^{n-1}(-a^{-1}q^{-j})(1-aq^j) = (-a^{-1})^n q^{-\binom{n}{2}}(a;q)_n.\end{aligned}$$

(1.2.32)

$$(a;q)_{n-k} = (a;q)_n(aq^n;q)_{-k} = \frac{(a;q)_n}{(aq^{n-k};q)_k}.$$

And

$$\begin{aligned}(aq^{n-k};q)_k &= \prod_{j=0}^{k-1}(1-aq^{n-k+j}) = \prod_{j=1}^{k}(1-aq^{n-j})\\ &= \prod_{j=1}^{k}(-aq^{n-j})(1-a^{-1}q^{j-n}) = (-a)^k q^{kn-\binom{k}{2}-k}(a^{-1}q^{1-n};q)_k\\ &= (-q/a)^{-k} q^{kn-\binom{k}{2}}(a^{-1}q^{1-n};q)_k,\end{aligned}$$

thus

$$(a;q)_{n-k} = \frac{(a;q)_n}{(a^{-1}q^{1-n};q)_k}(-q/a)^k q^{\binom{k}{2}-kn}.$$

(1.2.33)

For $n, k \in \mathbb{Z}_+$, See exercise 1.1(ii) on page 1. For $n, k \in \mathbb{Z}$, identity (1.2.33) may be verified by using repeated applications of identity (1.2.30).

$$\begin{aligned}(a;q)_{n+k} &= \frac{(a;q)_\infty}{(aq^{n+k};q)_\infty} = (a;q)_n \frac{(a;q)_\infty}{(a;q)_n(aq^{n+k})_\infty} \\ &= (a;q)_n \frac{(aq^n;q)_\infty}{(aq^{n+k};q)_\infty} \\ &= (a;q)_n(aq^n;q)_k.\end{aligned}$$

By the same argument, $(a;q)_{n+k} = (a;q)_k(aq^k;q)_n$.

(1.2.34)

See exercise 1.1(iii) on page 1. The argument is extended to negative integers by identity (1.2.33).

(1.2.35)

By (1.2.33),

$$(a;q)_{k+(n-k)} = (a;q)_k(aq^k;q)_{n-k} \iff (aq^k;q)_{n-k} = \frac{(a;q)_n}{(a;q)_k}.$$

(1.2.36)

By (1.2.33),

$$(a;q)_{2k+n-k} = (a;q)_{2k}(aq^{2k};q)_{n-k} \iff (aq^{2k};q)_{n-k} = \frac{(a;q)_{n+k}}{(a;q)_{2k}}.$$

Therefore

$$(aq^{2k};q)_{n-k} = \frac{(a;q)_n(aq^n;q)_k}{(a;q)_{2k}}.$$

(1.2.37)

$$\begin{aligned}(q^{-n};q)_k &= \prod_{j=0}^{k-1}(1-q^{j-n}) = \prod_{j=0}^{k-1}(-q^{j-n})(1-q^{n-j})\\ &= (-1)^k q^{\binom{k}{2}-nk}\prod_{j=0}^{k-1}(1-q^{n-j})\\ &= (-1)^k q^{\binom{k}{2}-nk}(1-q^n)(1-q^{n-1})\cdots(1-q^{n-(k-1)})\\ &= (-1)^k q^{\binom{k}{2}-nk}\frac{(1-q)(1-q^2)\cdots(1-q^{n-k+1})\cdots(1-q^n)}{(1-q)(1-q^2)\cdots(1-q^{n-k})}\\ &= (-1)^k q^{\binom{k}{2}-nk}\frac{(q;q)_n}{(q;q)_{n-k}}\end{aligned}$$

(1.2.38)

See exercise 1.1(iv) on page 2.

(1.2.39)

$$\begin{aligned}(a;q)_{2n} &= (1-a)(1-aq)(1-aq^2)(1-aq^3)\cdots(1-aq^{2n-2})(1-aq^{2n-1})\\ &= (1-a)(1-aq^2)\cdots(1-aq^{2n-2})\cdot(1-aq)(1-aq^3)\cdots(1-aq^{2n-1})\\ &= (a;q^2)_n(aq;q^2)_n\end{aligned}$$

(1.2.40)

$$\begin{aligned}(a^2;q^2)_n &= (1-a^2)(1-a^2q^2)\cdots(1-a^2q^{2n-2})\\ &= (1-a)(1+a)(1-aq)(1+aq)\cdots(1-aq^{n-1})(1+aq^{n-1})\\ &= (a;q)_n(-a;q)_n\end{aligned}$$

(i)

$$\begin{aligned}(aq^{-n};q)_n &= \prod_{j=0}^{n-1}(1-aq^{j-n}) = \prod_{j=1}^{n}(1-aq^{-j})\\ &= \prod_{j=1}^{n}(-aq^{-j})(1-a^{-1}q^j) = (-a)^n q^{-\binom{n}{2}-n}(q/a;q)_n\\ &= (q/a;q)_n\left(-\frac{a}{q}\right)^n q^{-\binom{n}{2}}.\end{aligned}$$

(ii)

$$
\begin{aligned}
(aq^{-k-n};q)_n &= \prod_{j=0}^{n-1}(1-aq^{-(k+n)+j}) = \prod_{j=1}^{n}(1-aq^{-(k+j)}) \\
&= \prod_{j=1}^{n}(-aq^{-(k+j)})(1-a^{-1}q^{k+j}) \\
&= (1-a^{-1}q^{k+1})(1-a^{-1}q^{k+2})\cdots(1-a^{-1}q^{k+n})\left(-\frac{a}{q}\right)^n q^{-kn-\binom{n}{2}} \\
&= \frac{(1-q/a)(1-q^2/a)\cdots(1-q^{n+k}/a)}{(1-q/a)(1-q^2/a)\cdots(1-q^k/a)}\left(-\frac{a}{q}\right)^n q^{-kn-\binom{n}{2}} \\
&= \frac{(q/a;q)_{n+k}}{(q/a;q)_k}\left(-\frac{a}{q}\right)^n q^{-kn-\binom{n}{2}}.
\end{aligned}
$$

(iii)

See exercise 1.1(v) on page 2.

References

[1] George Gasper. Lecture notes for an introductory minicourse on q-series, September 1995.

[2] George Gasper and Mizan Rahman. *Basic Hypergeometric Series*. Encyclopedia of Mathematics and Its Applications. Cambridge University Press, Cambridge, 2 edition, 2004.