

Exercises with q-Shifted Factorials

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Lecture Notes For An Introductory Minicourse on q-Series

Exercise 1.1

Verify the inversion identity.

Note that $0 + 1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2} = \binom{n}{2}$.

$$\begin{aligned}(a; q)_n &= (1-a)(1-aq)(1-aq^2) \cdots (1-aq^{n-1}) \\ &= (1-a)(q^{-1}-a)(q^{-2}-a) \cdots (q^{-(n-1)}-a) q^{\binom{n}{2}} \\ &= (1-a^{-1})(1-a^{-1}q^{-1}) \cdots (1-a^{-1}q^{-(n-1)}) (-a)^n q^{\binom{n}{2}} \\ &= (a^{-1}; q^{-1})_n (-a)^n q^{\binom{n}{2}}\end{aligned}$$

In parts (ii) through (v), k and n are nonnegative integers.

(i)

$$\begin{aligned}(a; q)_n &= (1-a)(1-aq) \cdots (1-aq^{n-1}) \\ &= \frac{(1-a)(1-aq) \cdots (1-aq^{n-1})(1-aq^n) \cdots}{(1-aq^n)(1-aq^{n+1}) \cdots} \\ &= \frac{(a; q)_\infty}{(aq^n; q)_\infty}\end{aligned}$$

(ii)

$$\begin{aligned}(a; q)_{n+k} &= (1-a) \cdots (1-aq^{n-1}) \cdot (1-aq^n) \cdots (1-aq^{n+k-1}) \\ &= (a; q)_n (aq^n; q)_k \\ &= (1-a) \cdots (1-aq^{k-1}) \cdot (1-aq^k) \cdots (1-aq^{k+n-1}) \\ &= (a; q)_k (aq^k; q)_n\end{aligned}$$

(iii)

$$(aq^n; q)_k = \frac{(a; q)_n (aq^n; q)_k}{(a; q)_n} = \frac{(a; q)_k (aq^k; q)_n}{(a; q)_n}$$

(iv)

By identity (ii),

$$(a; q)_{k-n} = (a; q)_{-n} (aq^{-n}; q)_k = (a; q)_k (aq^k; q)_{-n}.$$

We begin by writing

$$(aq^{-n}; q)_k = \frac{(a; q)_{k-n}}{(a; q)_{-n}} = \frac{(a; q)_k (aq^k; q)_{-n}}{(a; q)_{-n}}.$$

Using the definition $(a; q)_{-n} = (aq^{-n}; q)_n^{-1}$, we simplify this to

$$(aq^{-n}; q)_k = \frac{(a; q)_k (aq^{-n}; q)_n}{(aq^{k-n}; q)_n}.$$

We can simplify further:

$$\begin{aligned} (aq^{-n}; q)_n &= \prod_{j=0}^{n-1} (1 - aq^{-n} q^j) = \prod_{j=1}^n (1 - aq^{-j}) \\ &= \prod_{j=1}^n (-aq^{-j})(1 - a^{-1} q^j) = (-a)^n q^{-\binom{n+1}{2}} (a^{-1} q; q)_n, \end{aligned}$$

and

$$\begin{aligned} (aq^{k-n}; q)_n &= \prod_{j=0}^{n-1} (1 - aq^{k-n} q^j) = \prod_{j=1}^n (1 - aq^{k-j}) \\ &= \prod_{j=1}^n (-aq^{k-j})(1 - a^{-1} q^{j-k}) = (-a)^n q^{kn} q^{-\binom{n+1}{2}} (a^{-1} q^{1-k}; q)_n. \end{aligned}$$

Thus

$$\begin{aligned} (aq^{-n}; q)_k &= \frac{(a; q)_k (-a)^n q^{-\binom{n+1}{2}} (a^{-1} q; q)_n}{(-a)^n q^{kn} q^{-\binom{n+1}{2}} (a^{-1} q^{1-k}; q)_n} \\ &= \frac{(a; q)_k (a^{-1} q; q)_n}{(a^{-1} q^{1-k}; q)_n} q^{-kn}. \end{aligned}$$

(v)

$$\begin{aligned} \frac{1 - aq^{2n}}{1 - a} &= \frac{(aq^2; q^2)_n}{(a; q^2)_n} = \frac{(1 - aq^2) \cdots (1 - aq^{2n})}{(1 - a) \cdots (1 - aq^{2n-2})} \\ &= \frac{(1 - a^{\frac{1}{2}} q)(1 + a^{\frac{1}{2}} q) \cdots (1 - a^{\frac{1}{2}} q^n)(1 + a^{\frac{1}{2}} q^n)}{(1 - a^{\frac{1}{2}})(1 + a^{\frac{1}{2}}) \cdots (1 - a^{\frac{1}{2}} q^{n-1})(1 + a^{\frac{1}{2}} q^{n-1})} \\ &= \frac{(qa^{\frac{1}{2}}; q)_n (-qa^{\frac{1}{2}}; q)_n}{(a^{\frac{1}{2}}; q)_n (-a^{\frac{1}{2}}; q)_n}. \end{aligned}$$

Basic Hypergeometric Series

Exercise 1.1

Verify the identities (1.2.30)-(1.2.40); k and n are integers.

(1.2.30)

In the case that n is a positive integer, See exercise 1.1(i) on [page 1](#). If $n = -m$, $m \in \mathbb{Z}_+$ then

$$(a; q)_{-m} = \frac{1}{(aq^{-m}; q)_m} = \frac{(a; q)_\infty}{(aq^{-m}; q)_\infty}.$$

(1.2.31)

Note that $0 + 1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2} = \binom{n}{2}$.

$$\begin{aligned} (a^{-1}q^{1-n}; q)_n &= \prod_{j=0}^{n-1} (1 - a^{-1}q^{1-n+j}) = \prod_{j=0}^{n-1} (1 - a^{-1}q^{-j}) \\ &= \prod_{j=0}^{n-1} (-a^{-1}q^{-j})(1 - aq^j) = (-a^{-1})^n q^{-\binom{n}{2}} (a; q)_n. \end{aligned}$$

(1.2.32)

$$(a; q)_{n-k} = (a; q)_n (aq^n; q)_{-k} = \frac{(a; q)_n}{(aq^{n-k}; q)_k}.$$

And

$$\begin{aligned} (aq^{n-k}; q)_k &= \prod_{j=0}^{k-1} (1 - aq^{n-k+j}) = \prod_{j=1}^k (1 - aq^{n-j}) \\ &= \prod_{j=1}^k (-aq^{n-j})(1 - a^{-1}q^{j-n}) = (-a)^k q^{kn - \binom{k}{2} - k} (a^{-1}q^{1-n}; q)_k \\ &= (-q/a)^{-k} q^{kn - \binom{k}{2}} (a^{-1}q^{1-n}; q)_k, \end{aligned}$$

thus

$$(a; q)_{n-k} = \frac{(a; q)_n}{(a^{-1}q^{1-n}; q)_k} (-q/a)^k q^{\binom{k}{2} - kn}.$$

(1.2.33)

For $n, k \in \mathbb{Z}_+$, See exercise 1.1(ii) on [page 1](#). For $n, k \in \mathbb{Z}$, identity (1.2.33) may be verified by using repeated applications of identity (1.2.30).

$$\begin{aligned}
(a; q)_{n+k} &= \frac{(a; q)_\infty}{(aq^{n+k}; q)_\infty} = (a; q)_n \frac{(a; q)_\infty}{(a; q)_n (aq^{n+k})_\infty} \\
&= (a; q)_n \frac{(aq^n; q)_\infty}{(aq^{n+k}; q)_\infty} \\
&= (a; q)_n (aq^n; q)_k.
\end{aligned}$$

By the same argument, $(a; q)_{n+k} = (a; q)_k (aq^k; q)_n$.

(1.2.34)

See exercise 1.1(iii) on [page 1](#). The argument is extended to negative integers by identity (1.2.33).

(1.2.35)

By (1.2.33),

$$(a; q)_{k+(n-k)} = (a; q)_k (aq^k; q)_{n-k} \iff (aq^k; q)_{n-k} = \frac{(a; q)_n}{(a; q)_k}.$$

(1.2.36)

By (1.2.33),

$$(a; q)_{2k+n-k} = (a; q)_{2k} (aq^{2k}; q)_{n-k} \iff (aq^{2k}; q)_{n-k} = \frac{(a; q)_{n+k}}{(a; q)_{2k}}.$$

Therefore

$$(aq^{2k}; q)_{n-k} = \frac{(a; q)_n (aq^n; q)_k}{(a; q)_{2k}}.$$

(1.2.37)

$$\begin{aligned}
(q^{-n}; q)_k &= \prod_{j=0}^{k-1} (1 - q^{j-n}) = \prod_{j=0}^{k-1} (-q^{j-n})(1 - q^{n-j}) \\
&= (-1)^k q^{\binom{k}{2} - nk} \prod_{j=0}^{k-1} (1 - q^{n-j}) \\
&= (-1)^k q^{\binom{k}{2} - nk} (1 - q^n)(1 - q^{n-1}) \cdots (1 - q^{n-(k-1)}) \\
&= (-1)^k q^{\binom{k}{2} - nk} \frac{(1 - q)(1 - q^2) \cdots (1 - q^{n-k+1}) \cdots (1 - q^n)}{(1 - q)(1 - q^2) \cdots (1 - q^{n-k})} \\
&= (-1)^k q^{\binom{k}{2} - nk} \frac{(q; q)_n}{(q; q)_{n-k}}
\end{aligned}$$

(1.2.38)

See exercise 1.1(*iv*) on [page 2](#).

(1.2.39)

$$\begin{aligned}
(a; q)_{2n} &= (1 - a)(1 - aq)(1 - aq^2)(1 - aq^3) \cdots (1 - aq^{2n-2})(1 - aq^{2n-1}) \\
&= (1 - a)(1 - aq^2) \cdots (1 - aq^{2n-2}) \cdot (1 - aq)(1 - aq^3) \cdots (1 - aq^{2n-1}) \\
&= (a; q^2)_n (aq; q^2)_n
\end{aligned}$$

(1.2.40)

$$\begin{aligned}
(a^2; q^2)_n &= (1 - a^2)(1 - a^2 q^2) \cdots (1 - a^2 q^{2n-2}) \\
&= (1 - a)(1 + a)(1 - aq)(1 + aq) \cdots (1 - aq^{n-1})(1 + aq^{n-1}) \\
&= (a; q)_n (-a; q)_n
\end{aligned}$$

(i)

$$\begin{aligned}
(aq^{-n}; q)_n &= \prod_{j=0}^{n-1} (1 - aq^{j-n}) = \prod_{j=1}^n (1 - aq^{-j}) \\
&= \prod_{j=1}^n (-aq^{-j})(1 - a^{-1}q^j) = (-a)^n q^{-\binom{n}{2} - n} (q/a; q)_n \\
&= (q/a; q)_n \left(-\frac{a}{q}\right)^n q^{-\binom{n}{2}}.
\end{aligned}$$

(ii)

$$\begin{aligned}
(aq^{-k-n}; q)_n &= \prod_{j=0}^{n-1} (1 - aq^{-(k+n)+j}) = \prod_{j=1}^n (1 - aq^{-(k+j)}) \\
&= \prod_{j=1}^n (-aq^{-(k+j)})(1 - a^{-1}q^{k+j}) \\
&= (1 - a^{-1}q^{k+1})(1 - a^{-1}q^{k+2}) \cdots (1 - a^{-1}q^{k+n}) \left(-\frac{a}{q}\right)^n q^{-kn - \binom{n}{2}} \\
&= \frac{(1 - q/a)(1 - q^2/a) \cdots (1 - q^{n+k}/a)}{(1 - q/a)(1 - q^2/a) \cdots (1 - q^k/a)} \left(-\frac{a}{q}\right)^n q^{-kn - \binom{n}{2}} \\
&= \frac{(q/a; q)_{n+k}}{(q/a; q)_k} \left(-\frac{a}{q}\right)^n q^{-kn - \binom{n}{2}}.
\end{aligned}$$

(iii)

See exercise 1.1(v) on [page 2](#).

References

- [1] George Gasper. Lecture notes for an introductory minicourse on q-series, September 1995.
- [2] George Gasper and Mizan Rahman. *Basic Hypergeometric Series*. Encyclopedia of Mathematics and Its Applications. Cambridge University Press, Cambridge, 2 edition, 2004.